Exercise 11B: Solutions

1 Different angles may be used.

a
$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ \\ &\quad + \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

b
$$\begin{aligned}\cos 105^\circ &= \cos(45+60)^\circ\\ &= \cos 45^\circ \cos 60^\circ \\ &\quad - \sin 45^\circ \sin 60^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

2 Different angles may be used.

a
$$\begin{aligned}\sin 165^\circ &= \sin(180-15)^\circ\\ &= \sin 15^\circ\\ \sin 15^\circ &= \sin(45-30)^\circ\\ &= \sin 45^\circ \cos 30^\circ \\ &\quad - \cos 45^\circ \sin 30^\circ\end{aligned}$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

b
$$\begin{aligned}\tan 75^\circ &= \tan(45+30)^\circ\\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\\ &= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{3+2\sqrt{3}+1}{3-1} = 2+\sqrt{3}\end{aligned}$$

3 Different angles may be used.

a
$$\begin{aligned}\cos\frac{5\pi}{12} &= \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b
$$\begin{aligned}\sin\frac{11\pi}{12} &= \sin\left(\pi-\frac{\pi}{12}\right)\\ &= \sin\frac{\pi}{12}\end{aligned}$$

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

c
$$\begin{aligned}\tan\left(-\frac{\pi}{12}\right) &= \tan\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\\ &= \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{3}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{3}}\\ &= \frac{1-\sqrt{3}}{1+\sqrt{3}}\times\frac{1-\sqrt{3}}{1-\sqrt{3}}\\ &= \frac{1-2\sqrt{3}+3}{1-3}\\ &= -2+\sqrt{3}\end{aligned}$$

4
$$\begin{aligned}\cos^2 u &= 1-\sin^2 u\\ &= 1-\frac{144}{169}=\frac{25}{169}\\ \cos u &= \pm\frac{5}{13}\\ \cos^2 v &= 1-\sin^2 v\\ &= 1-\frac{9}{25}=\frac{16}{25}\\ \cos v &= \pm\frac{4}{5}\end{aligned}$$

$$\begin{aligned}\sin(u+v) &= \sin u\cos v+\cos u\sin v\\ &= \pm\frac{3}{5}\times\frac{5}{13}\pm\frac{4}{5}\times\frac{12}{13}\\ &= \frac{\pm15\pm48}{65}\end{aligned}$$

There are four possible answers:
$\frac{63}{65}, \frac{33}{65}, -\frac{33}{65}, -\frac{63}{65}$

5 a

$$\begin{aligned}\sin\left(\theta+\frac{\pi}{6}\right) &= \sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\\ &= \frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta\end{aligned}$$

b

$$\begin{aligned}\cos\left(\pi-\frac{\pi}{4}\right) &= \cos\phi\cos\frac{\pi}{4}+\sin\phi\sin\frac{\pi}{4}\\ &= \frac{1}{\sqrt{2}}\cos\phi+\frac{1}{\sqrt{2}}\sin\phi\\ &= \frac{1}{\sqrt{2}}(\cos\phi+\sin\phi)\end{aligned}$$

c

$$\begin{aligned}\tan\left(\theta+\frac{\pi}{6}\right) &= \frac{\tan\theta+\tan\frac{\pi}{3}}{1-\tan\theta\tan\frac{\pi}{3}}\\ &= \frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}\end{aligned}$$

d

$$\begin{aligned}\sin\left(\theta-\frac{\pi}{4}\right) &= \sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}\\ &= \frac{1}{\sqrt{2}}\sin\theta-\frac{1}{\sqrt{2}}\cos\theta\\ &= \frac{1}{\sqrt{2}}(\sin\theta-\cos\theta)\end{aligned}$$

6 a $\sin(v+(u-v)) = \sin u$

b $\cos((u+v)-v) = \cos u$

7

$$\begin{aligned}\cos^2\theta &= 1-\sin^2\theta\\ &= 1-\frac{9}{25}=\frac{16}{25}\\ \cos\theta &= -\frac{4}{5}\end{aligned}$$

(Since $\cos\theta < 0$)

$$\begin{aligned}\sin^2\phi &= 1-\cos^2\phi\\ &= 1-\frac{25}{169}=\frac{144}{169}\\ \sin\phi &= \frac{12}{13}\end{aligned}$$

(Since $\sin\theta > 0$)

a

$$\begin{aligned}\cos 2\phi &= \cos^2\phi-\sin^2\phi\\ &= \frac{25}{169}-\frac{144}{169}\\ &= -\frac{119}{169}\end{aligned}$$

b $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times -\frac{3}{5} \times -\frac{4}{5}$$

$$= \frac{24}{25}$$

c $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{-3}{-4} = \frac{3}{4}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{\frac{3}{2}}{1 - \frac{9}{16}}$$

$$= \frac{3}{2} \times \frac{16}{7}$$

$$= \frac{24}{7}$$

d $\sec 2\phi = \frac{1}{\cos 2\phi}$

$$= -\frac{169}{119}$$

e $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= -\frac{3}{5} \times -\frac{5}{13} + -\frac{4}{5} \times \frac{12}{13}$$

$$= \frac{14 - 48}{65}$$

$$= -\frac{33}{65}$$

f $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= -\frac{4}{5} \times -\frac{5}{13} + -\frac{3}{5} \times \frac{12}{13}$$

$$= \frac{20 - 36}{65}$$

$$= -\frac{16}{65}$$

g $\operatorname{cosec}(\theta + \phi) = \frac{1}{\sin(\theta + \phi)}$

$$= -\frac{65}{33}$$

h $\cot 2\theta = \frac{1}{\tan 2\theta}$

$$= \frac{7}{24}$$

8 **a** $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

$= \left(\dfrac{4}{3} + \dfrac{5}{12}\right) \div \left(1 - \dfrac{4}{3} \times \dfrac{5}{12}\right)$

$= \dfrac{21}{12} \div \dfrac{4}{9}$

$= \dfrac{21}{12} \times \dfrac{9}{4}$

$= \dfrac{63}{16}$

b $\tan 2u = \dfrac{2\tan u}{1 - \tan^2 u}$

$= \dfrac{\frac{8}{3}}{1 - \frac{16}{9}}$

$= \dfrac{8}{3} \times \dfrac{9}{-7}$

$= -\dfrac{24}{7}$

c $\sec^2 u = 1 + \tan^2 u$

$= 1 + \dfrac{16}{9} = \dfrac{25}{9}$

$\cos^2 u = \dfrac{9}{25}$

$\cos u = \dfrac{3}{5}$ (since\ u is acute)

$\sec^2 v = 1 + \tan^2 v$

$= 1 + \dfrac{25}{144} = \dfrac{169}{144}$

$\cos^2 v = \dfrac{144}{169}$

$\cos v = \dfrac{12}{13}$ (since\ v is acute)

$\cos(u-v) = \cos u \cos v + \sin u \sin v$

$= \dfrac{3}{5} \times \dfrac{12}{13} + \dfrac{4}{5} \times \dfrac{5}{13}$

$= \dfrac{56}{65}$

d $\sin 2u = 2\sin u \cos u$

$= 2 \times \dfrac{4}{5} \times \dfrac{3}{5}$

$= \dfrac{24}{25}$

9 $\cos\alpha = -\dfrac{4}{5}$

$\cos^2\beta = 1 - \sin^2\beta$

$= 1 - \dfrac{576}{625} = \dfrac{29}{625}$

$\cos\beta = -\dfrac{7}{25}$

$\cos^2\alpha = 1 - \sin^2\alpha$

$= 1 - \dfrac{9}{25} = \dfrac{16}{25}$

a $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$= \frac{16}{25} - \frac{9}{25}$$

$$= \frac{7}{25}$$

b $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{3}{5} \times -\frac{7}{25} - -\frac{4}{5} \times \frac{24}{25}$$

$$= \frac{75}{125} = \frac{3}{5}$$

c

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$= -\frac{3}{4}$$

$$\tan\beta = \frac{\sin\beta}{\cos\beta}$$

$$= -\frac{24}{7}$$

$$\tan\ (\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{-\frac{3}{4} + -\frac{24}{7}}{1 - -\frac{3}{4} \times \frac{24}{7}}$$

$$= -\frac{117}{28} \times -\frac{7}{11}$$

$$= \frac{117}{44}$$

d $\sin 2\beta = 2\sin\beta\cos\beta$

$$= 2 \times \frac{7}{25} \times -\frac{24}{25}$$

$$= -\frac{336}{625}$$

10a $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times -\frac{\sqrt{3}}{2} \times \frac{1}{2}$$

$$= -\frac{\sqrt{3}}{2}$$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \frac{1}{4} - \frac{3}{4}$$

$$= -\frac{1}{2}$$

11a $(\sin\theta - \cos\theta)^2 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$

$$= 1 - \sin 2\theta$$

b $\sin^4\theta - \cos^4\theta = (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$

$$= \cos 2\theta \times 1$$

$$= \cos 2\theta$$

12

a $$\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)=\sqrt{2}\left(\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}\right)$$
$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta-\frac{1}{\sqrt{2}}\cos\theta\right)$$
$$=\sin\theta-\cos\theta$$

b $$\cos\left(\theta-\frac{\pi}{3}\right)=\cos\theta\cos\frac{\pi}{3}+\sin\theta\sin\frac{\pi}{3}$$
$$=\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta$$
$$\cos\left(\theta+\frac{\pi}{3}\right)=\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta$$

Add the last two equations:

$$\cos\left(\theta-\frac{\pi}{3}\right)+\cos\left(\theta+\frac{\pi}{3}\right)=\cos\theta$$

c $$\tan\left(\theta+\frac{\pi}{4}\right)\tan\left(\theta-\frac{\pi}{4}\right)=\frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}\times\frac{\tan\theta-\tan\frac{\pi}{4}}{1+\tan\theta\tan\frac{\pi}{4}}$$
$$=\frac{\tan\theta+1}{1-\tan\theta}\times\frac{\tan\theta-1}{1+\tan\theta}$$
$$=-1$$

d $$\cos\left(\theta+\frac{\pi}{6}\right)=\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}$$
$$=\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta$$
$$\sin\left(\theta+\frac{\pi}{3}\right)=\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}$$
$$=\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta$$

Add the two equations:

$$\cos\left(\theta+\frac{\pi}{6}\right)+\sin\left(\theta+\frac{\pi}{3}\right)=\sqrt{3}\cos\theta$$

e $$\tan\left(\theta+\frac{\pi}{4}\right)=\frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}$$
$$=\frac{\tan\theta+1}{1-\tan\theta}$$
$$=\frac{1+\tan\theta}{1-\tan\theta}$$

f $$\frac{\sin(u+v)}{\cos u\cos v}=\frac{\sin u\cos v+\cos u\sin v}{\cos u\cos v}$$
$$=\frac{\sin u\cos v}{\cos u\cos v}+\frac{\cos u\sin v}{\cos u\cos v}$$
$$=\tan u+\tan v$$

g $$\frac{\sin(u+v)}{\sin(u-v)} = \frac{\sin u \cos v + \cos u \sin v}{\sin u \cos v - \cos u \sin v}$$

Divide numerator and denominator by $\cos u \cos v$.

$$\frac{\sin(u+v)}{\sin(u-v)} = \frac{\tan u + \tan v}{\tan u - \tan v}$$

h $$\begin{aligned}\cos 2\theta + 2\sin^2\theta &= \cos^2\theta - \sin\theta + 2\sin^2\theta \\ &= \cos^2\theta + \sin^2\theta \\ &= 1\end{aligned}$$

i $$\begin{aligned}\sin 4\theta &= \sin(2 \times 2\theta) \\ &= 2\sin 2\theta \cos 2\theta \\ &= 2 \times 2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) \\ &= 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta\end{aligned}$$

j $$\begin{aligned}\frac{1-\sin 2\theta}{\sin\theta - \cos\theta} &= \frac{1-\sin 2\theta}{\sin\theta - \cos\theta} \times \frac{\sin\theta - \cos\theta}{\sin\theta - \cos\theta} \\ &= \frac{(1-\sin 2\theta)(\sin\theta - \cos\theta)}{\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta} \\ &= \frac{(1-\sin 2\theta)(\sin\theta - \cos\theta)}{1 - 2\sin\theta\cos\theta} \\ &= \frac{(1-\sin 2\theta)(\sin\theta - \cos\theta)}{1 - \sin 2\theta} \\ &= \sin\theta - \cos\theta\end{aligned}$$